LOCAL STRUCTURE OF PRINCIPALLY POLARIZED STABLE LAGRANGIAN FIBRATIONS

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Abstract. A holomorphic Lagrangian fibration is stable if the characteristic cycles of the singular fibers are of type $I_m, 1 \leq m < \infty$, or A_∞. We will give a complete description of the local structure of a stable Lagrangian fibration when it is principally polarized. In particular, we give an explicit form of the period map of such a fibration and conversely, for a period map of the described type, we construct a principally polarized stable Lagrangian fibration with the given period map. This enables us to give a number of examples exhibiting interesting behavior of the characteristic cycles.

1. Introduction

For a holomorphic symplectic manifold (M, ω), i.e., a $2n$-dimensional complex manifold with a holomorphic symplectic form $\omega \in H^0(M, \Omega^2_M)$, a proper flat morphism $f : M \to B$ over an n-dimensional complex manifold B is called a (holomorphic) Lagrangian fibration if all smooth fibers are Lagrangian submanifolds of M. The discriminant $D \subset B$, i.e., the set of critical values of f, is a hypersurface if it is non-empty. In [HO1], the structure of the singular fiber of f at a general point $b \in D$ was studied. By introducing the notion of characteristic cycles, [HO1] shows that the structure of such a singular fiber can be described in a manner completely parallel to Kodaira's classification ([Kd], see also V. 7 in [BHPV]) of singular fibers of elliptic fibrations. Furthermore, to study the multiplicity of the singular fibers, [HO2] generalized the stable reduction theory of elliptic fibrations (cf. V.10 in [BHPV]), explicitly describing how arbitrary singular fiber over a general point of D can be transformed to a stable singular fiber, a singular fiber of particularly simple type. These results exhibit that the theory of general singular fibers of a holomorphic Lagrangian fibration gives a very natural generalization of Kodaira's theory of elliptic fibrations.

The current work is yet another manifestation of this principle. An important part of Kodaira's theory is the study of the asymptotic behavior of the elliptic modular function of a given elliptic fibration near a singular fiber. As a generalization of this we will study the asymptotic behavior of the periods of the abelian fibers near a general singular fiber of a holomorphic Lagrangian fibration. Here we need to make two additional assumptions on the Lagrangian fibration.

First, we will assume that the singular fibers are of stable type, i.e., its characteristic cycles are of type $I_k, 0 \leq k \leq \infty$ (I_∞ meaning A_∞). As explained above, any general singular fiber can be transformed into this form by the stable reduction ([HO2], Section 4).

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The second assumption we will make is that the Lagrangian fibration is *principally polarized*, in the sense explained in Definition 3.1. This condition is satisfied if there exists an f-ample line bundle on $M \setminus f^{-1}(D)$ whose restriction on smooth fibers give principal polarizations on the abelian varieties. This assumption is rather restrictive compared with the setting of [HO1] and [HO2], where the only assumption was that the fibers of f are of Fujiki class.

We believe that understanding the structure of Lagrangian fibration under these assumptions is essential for the study of general cases. Since this special case already requires substantial care and already provides many interesting examples (see Section 5), we restrict our attention to it in this paper and leave the general cases to future study.

The main result of this paper is the following.

Theorem 1.1. *Let $f : M \to B$ be a principally polarized stable Lagrangian fibration (cf. Definition 2.1 and Definition 3.1). Then at a general point $b \in D$ of the discriminant, there exists a coordinate system $(z_1, \ldots, z_n)$ with D defined by $z_n = 0$, such that for a suitable choice of an integral frame of the local system $R^1 f_* \mathbb{Z}$ on $B \setminus D$, the period matrices have the form*

$$\theta^i_j = \frac{\partial^2 \Psi}{\partial z_i \partial z_j} \text{ for } (i,j) \neq (n,n) \text{ and } \theta^n_n = \frac{\partial^2 \Psi}{\partial z_n \partial z_n} + \frac{\ell}{2\pi\sqrt{-1}} \log z_n$$

where Ψ is a holomorphic function in $z_1, \ldots, z_n$. Here ℓ is the number of irreducible components of a general singular fiber.

Conversely, given any germ of holomorphic function $\Psi(z_1, \ldots, z_n)$ such that $\mathrm{Im}(\theta^i_j) > 0$, there exists a principally polarized stable Lagrangian fibration whose period matrices are of the above form.

That the period matrix of Lagrangian fibration is the Hessian of a potential function is a well-known consequence of the action-angle variables (cf. [DM]). The logarithmic behavior of the multi-valued part reflects the stability assumption on the singular fiber. The novelty in Theorem 1.1 lies in the choice of the variable z_n through which these two aspects are intertwined. The existence of z_n follows from the fact proved in Proposition 3.13 that the characteristic foliation accounts for the degenerate part of the polarization restricted to the fixed part of the monodromy. The proof of this uses a version of the Monodromy Theorem from the theory of the degeneration of Hodge structures and the topological property of the stable singular fiber.

The converse direction in Theorem 1.1 is shown by explicitly constructing a principally polarized stable Lagrangian fibration from a given potential function $\Psi(z)$ imaginary part of whose Hessian matrix is positive definite. This part is a sort of generalization of Nakamura's construction [Na] of toroidal degeneration of principally polarized abelian varieties over 1-dimensional small disk. Using our construction, we shall give a concrete 4-dimensional example of principally polarized stable Lagrangian fibration in which the types of characteristic cycles of singular fibers change fiber by fiber, too. To our knowledge, such an example has not been noticed previously. In fact, most of the previous constructions of singular fibers of Lagrangian fibrations have used product construction from elliptic fibrations.

2. Stable Lagrangian fibrations

Definition 2.1. A *Lagrangian fibration* is a proper flat morphism $f : M \to B$ from a holomorphic symplectic manifold (M, ω) of dimension $2n$ to a complex manifold B of dimension n such that the smooth locus of each fiber is a Lagrangian submanifold of M. The *discriminant* $D \subset B$ is the set of the critical values of f, which is a hypersurface in B if it is non-empty. *Throughout this paper, we assume that D is non-empty.* We say that f is a *stable Lagrangian fibration* if $D \subset B$ is a submanifold and each singular fiber $f^{-1}(b), b \in D$, is stable, i.e., it is reduced and the characteristic cycle in the sense of [HO1] is of type $I_k, 1 \leq k \leq \infty$. By the description in [HO1], this is equivalent to saying that $f^{-1}(b)$ is reduced and its normalization is a disjoint union of a finite number of compact complex manifolds $Y^1, \dots, Y^\ell$ for some positive integer ℓ such that

(i) each Y^i is a $\mathbb{P}^1$-bundle over an $(n-1)$-dimensional complex torus A^i whose fibers are sent to characteristic leaves of $f^{-1}(b)$ in the sense of [HO1], i.e., for a defining function $h \in \mathcal{O}(B)$ of the divisor D, the Hamiltonian vector field $\iota_\omega(f^*dh)$, where $\iota_\omega : \Omega^1_M \to T(M)$ is the vector bundle isomorphism induced by ω, is tangent to the image of the fibers in M;
(ii) there exist submanifolds $S^i_1, S^i_2 \subset Y^i$, with $S^i_1 \neq S^i_2$ except possibly when $\ell = 1, 2$, such that $S^i_1 \cup S^i_2$ is a 2-to-1 unramified cover of A^i under the $\mathbb{P}^1$-bundle projection;
(iii) the normalization $\nu : \bigcup Y^i \to f^{-1}(b)$ is obtained by the identification via a collection of biholomorphic morphisms $g_i : S^i_2 \to S^{i+1}_1$ for $1 \leq i \leq \ell - 1$ and $g_\ell : S^\ell_2 \to S^1_1$ with the additional requirement $g_1 = g_2^{-1}$ if $S^1_1 = S^1_2$ and $S^2_1 = S^2_2$ for $\ell = 2$.

A maximal connected union of the $\mathbb{P}^1$-fibers in (i) under the identification in (iii) is called a *characteristic cycle*. A characteristic cycle can be either of finite type (I_m-type, $1 \leq m < \infty$) or of infinite type A_∞, which we also denote by I_∞.

Recall (cf. [HO2] Section 4) that in a neighborhood of a general singular fiber, any Lagrangian fibration whose fibers are of Fujiki class can be transformed to a stable Lagrangian fibration by certain explicitly given bimeromorphic modifications and branched covering. We will be interested in the local property of the fibration at a point of D. Thus we will make the following

(*Assumption*) $D \subset B$ is the germ of a smooth hypersurface in an n-dimensional complex manifold and the fundamental group $\pi_1(B \setminus D)$ is cyclic.

The following is immediate from Proposition 2.2 of [HO1].

Proposition 2.2. *Given a stable Lagrangian fibration, we can assume that there exists an action of the complex Lie group $\mathbb{C}^{n-1}$ on M preserving the fibers and the symplectic form such that S^i_1, S^i_2 are orbits of this action for all $1 \leq i \leq \ell$. This action of $\mathbb{C}^{n-1}$ on Y^i descends to the translation action on A^i. The patching biholomorphisms g_i in Definition 2.1 (iii) as well as the $\mathbb{P}^1$-bundle structure in (i) are equivariant under this action. In particular, if $S^1_1 = S^1_2$ (resp. $S^2_1 = S^2_2$), the Galois action of the double cover $S^1_1 \to A^1$ (resp. $S^2_1 \to A^2$) is by a translation on the torus S^1_1 (resp. S^2_1).*

Regarding the topology of the singular fiber $f^{-1}(b), b \in D$, we have the following.

Proposition 2.3. *In the setting of Definition 2.1, fix a component Y^1 of the normalization of $f^{-1}(b)$ and set*

$$Y_o := Y^1 \setminus (S_1^1 \cup S_2^1),$$

which is equipped with a $\mathbb{C}^*$-bundle structure $\varrho : Y_o \to A$ over a complex torus A of dimension $n-1$ coming from Definition 2.1 (i). Then there exists a (not necessarily holomorphic) continuous map $\mu : f^{-1}(b) \to A'$ to a complex torus A' isogenous to A such that when $j : Y_0 \to f^{-1}(b)$ is the natural inclusion and $\rho : Y_0 \to A \to A'$ is the composition of ϱ and an isogeny, $\mu \circ j$ is homotopic to ρ.

Proof. Let us use the notation introduced in Definition 2.1 (iii) for the description of the normalization morphism $\nu : \bigcup Y^i \to f^{-1}(b)$.

First, we consider the case $S_1^i \neq S_2^i$ for all $1 \leq i \leq \ell$. Define $\widetilde{f^{-1}(b)}$ as the variety obtained from $\bigcup Y^i$ with all the patching identification $g_1, \ldots, g_{\ell-1}$ such that the normalization factors through

$$\nu : \bigcup Y^i \to \widetilde{f^{-1}(b)} \to f^{-1}(b)$$

with the second arrow given by the identification via g_ℓ. When $\ell = 1$, $\widetilde{f^{-1}(b)} = Y^1$. The $\mathbb{C}^{n-1}$-action of Proposition 2.2 lifts to a $\mathbb{C}^{n-1}$-action on $\widetilde{f^{-1}(b)}$. The connected unions of the images of the $\mathbb{P}^1$-fibers of Y^i define finite chains of quasi-transversally intersecting $\mathbb{P}^1$'s in $\widetilde{f^{-1}(b)}$, which we call characteristic chains. Each characteristic chain intersects each S_1^i (resp. S_2^i), $1 \leq i \leq \ell$ at exactly one point, inducing a morphism $\widetilde{f^{-1}(b)} \to A''$ to a complex torus of dimension $n-1$ biholomorphic to A^i's. This determines a biholomorphism $\zeta : S_2^\ell \to S_1^1$. Fix a point $\alpha \in S_2^\ell$ and let $\beta = \zeta(\alpha) \in S_1^1$. For $t \in [0,1] \subset \mathbb{R}$, let $\tau_t : S_1^1 \to S_1^1$ be the translation by $t(\beta - g_\ell(\alpha))$.

Define a new family of biholomorphic morphisms $g_\ell^t : S_2^\ell \to S_1^1$ by $g_\ell^t = \tau_t \circ g_\ell$. Clearly, $g_\ell^0 = g_\ell$. We claim that $g_\ell^1 = \zeta$. In fact, $\zeta^{-1} \circ g_\ell^1$ is an automorphism of S_2^ℓ which fixes the point α. But both g_ℓ^1 and ζ must be equivariant under the $\mathbb{C}^{n-1}$-action of Proposition 2.2. Thus $\zeta^{-1} \circ g_\ell^1$ must be the identity map of S_2^ℓ, proving the claim.

Let $f^{-1}(b)^t$ be the variety obtained from $\widetilde{f^{-1}(b)}$ by identifying S_1^1 and S_2^ℓ via g_ℓ^t. Then

$$f^{-1}(b)^0 = f^{-1}(b)$$

and $f^{-1}(b)^1$ is homeomorphic to $f^{-1}(b)$. The $\mathbb{C}^{n-1}$-action descends to $f^{-1}(b)^t$ for each t as τ_t commutes with the $\mathbb{C}^{n-1}$-action. By abuse of terminology, we call the maximal connected unions of the images in $f^{-1}(b)^t$ of the characteristic chains as characteristic cycles of $f^{-1}(b)^t$. By the $\mathbb{C}^{n-1}$-action, we know that all characteristic cycles in $f^{-1}(b)^t$ are isomorphic. By our choice of α and β, there exists one finite characteristic cycle in $f^{-1}(b)^1$. Thus we get a morphism $\mu' : f^{-1}(b)^1 \to A'$ to some complex torus A' isogenous to A''. Define $\mu : f^{-1}(b) \to A'$ as the composition of μ' with the homeomorphism $f^{-1}(b) \to f^{-1}(b)^1$. It certainly satisfies the required property.

Now consider the case when $\ell = 1$ and $S_1^1 = S_2^1$. Set $\widetilde{f^{-1}(b)} = Y^1$ and define $\zeta : S_2^1 \to S_1^1$ as the Galois action of the double covering $S_1^1 \to A^1$ in Definition 2.1 (ii). Then the same argument as in the previous case applies.

Finally, consider the case when $\ell = 2$, $S_1^1 = S_2^1$ and $S_1^2 = S_2^2$. By Proposition 2.2, the Galois action on S_1^1 (resp. S_1^2) of the double cover over A^1 (resp. A^2) is given by a translation, say, by $\gamma_1 \in \mathbb{C}^{n-1}$ (resp. $\gamma_2 \in \mathbb{C}^{n-1}$). By the equivariance of $g_1 = g_2^{-1}$, for each $\alpha \in S_1^1$, we have $g_1(\gamma_1 \cdot \alpha) = \gamma_2 \cdot g_1(\alpha)$. Thus by the normalization morphism $\nu : Y^1 \cup Y^2 \to f^{-1}(b)$, a point $\alpha \in S_1^1$ is identified with $g_1(\alpha) \in S_1^2$, and the point $\gamma_1 \cdot \alpha \in S_1^1$, which lies in the $\mathbb{P}^1$-fiber through α, is identified with $\gamma_2 \cdot g_1(\alpha)$, which lies in the $\mathbb{P}^1$-fiber through $g_1(\alpha)$. Thus we get a morphism $\mu : f^{-1}(b) \to A'$ to an $(n-1)$-dimensional torus A' whose fiber is a union of two $\mathbb{P}^1$'s identified at two points. This μ satisfies the required property. $\square$

We have the generalization of the classical action-angle correspondence as follows.

Proposition 2.4. *Given a stable Lagrangian fibration $f : M \to B$, choose a Lagrangian section $\Sigma \subset M$ of f. Then we have a natural surjective unramified morphism $\Phi : T^*B \to M \setminus E$ where E is the union of the irreducible components of the fibers of f disjoint from Σ such that*

1. *$f \circ \Phi$ agrees with the natural projection $g : T^*B \to B$,*
2. *Φ sends the zero section of T^*B to Σ and*
3. *$\Phi^*\omega$ coincides with the standard symplectic form on T^*B.*

In particular, $\Gamma := \Phi^{-1}(\Sigma)$ is a Lagrangian submanifold (with many connected components) in T^*B. For each $b \in B$, $\Phi_b := \Phi|_{T_b^*(B)} : T_b^*(B) \to f^{-1}(b) \setminus E$ is the universal covering and $\Gamma_b := \Gamma \cap T_b^*(B)$ is naturally isomorphic to $H_1(f^{-1}(b) \setminus E, \mathbb{Z})$.

Proof. Over $B \setminus D$, this is just a holomorphic version (cf. Proposition 3.5 in [Hw]) of the classical action-angle correspondence as described in Section 44 of [GS]. The statement over D follows by the same argument as for the smooth fibers. In fact, for each $b \in B$, the vector group $T_b^*(B)$ acts on the fiber $f^{-1}(b)$ with n-dimensional orbits on the smooth locus of $f^{-1}(b)$ (cf. Proposition 3.3 in [Hw]). The morphism Φ_b is defined by taking the orbit map of the point $\Sigma \cap f^{-1}(b)$ under this action, which is a universal covering map for the smooth locus of the component of $f^{-1}(b)$ containing $\Sigma \cap f^{-1}(b)$. This shows (1) and (2). The proof of (3) is the same as that of Theorem 44.2 of [GS]. $\square$

Proposition 2.5. *Let $b \in D$. In the notation of Proposition 2.4, we can assume that the connected component of Γ containing each point of $\Gamma \cap T_b^*(B)$ is a Lagrangian section of $T^*(B) \to B$, i.e., a closed 1-form on B. Let $\Gamma' \subset \Gamma$ be the union of such sections of Γ over B. Then for each $s \in B \setminus D$, $\Gamma'_s := \Gamma' \cap T_s^*(B)$ is a sublattice of Γ_s satisfying $\Gamma_s/\Gamma'_s \cong \mathbb{Z}$.*

Proof. Since $f^{-1}(b) \setminus E$ is a $\mathbb{C}^*$-bundle over an $(n-1)$-dimensional torus, we see that $\Gamma \cap T_b^*(B)$ has rank $2n-1$. Thus Γ'_s has rank $2n-1$. It remains to show that Γ_s/Γ'_s is torsion-free. Suppose it has a k-torsion, $0 < k \in \mathbb{Z}$, i.e., there exists a point $\alpha \in \Gamma_s \setminus \Gamma'_s$ such that $k\alpha \in \Gamma'_s$. Let $\widetilde{k\alpha}$ be a closed 1-form given by the component of Γ' containing $k\alpha$. Then the closed 1-form $\tilde{\alpha} = \frac{1}{k}\widetilde{k\alpha}$ is also a component of Γ' containing α, which implies $\alpha \in \Gamma'_s$, a contradiction. $\square$

Proposition 2.6. *In the notation of Proposition 2.4, let Y_o be the fiber of $M \setminus E$ at a point $b \in D$. Let $\Phi_b : T_b^*(B) \to Y_o$ be the universal covering map and $\varrho : Y_o \to A$ be the $\mathbb{C}^*$-bundle*

over an $(n-1)$-dimensional torus. Let $\Upsilon \subset \Gamma_b = \Gamma'_b$ be the rank-1 sublattice corresponding to the kernel of

$$\varrho_* : H_1(Y_0, \mathbb{Z}) \to H_1(A, \mathbb{Z}).$$

Then for any $v \in \Gamma'_b \setminus \Upsilon$, there exists $\varpi \in H^1(f^{-1}(b), \mathbb{Z})$ such that $\langle \varpi, j_* v\rangle \neq 0$ where $j_* : H_1(Y_o, \mathbb{Z}) \to H_1(f^{-1}(b), \mathbb{Z})$ is induced by the inclusion $j : Y_o \subset f^{-1}(b)$.

Proof. Since $\varrho_*(v) \in H_1(A, \mathbb{Z})$ is non-zero, there exists $\varphi \in H^1(A, \mathbb{Z})$ such that $\langle \varphi, \varrho_*(v)\rangle \neq 0$. Let $\varpi = \mu^*\varphi$ where the map $\mu : f^{-1}(b) \to A$ is as defined in Proposition 2.3 satisfying $\mu \circ j = \varrho$. Then

$$\langle \varpi, j_*(v)\rangle = \langle \mu^*\varphi, j_*(v)\rangle = \langle j^*\mu^*\varphi, v\rangle = \langle \varrho^*\varphi, v\rangle = \langle \varphi, \varrho_*(v)\rangle \neq 0.$$

□

For a stable Lagrangian fibration $f : M \to B$, denote by Λ the local system on $B \setminus D$ defined by the lattice $\Lambda_s := H_1(f^{-1}(s), \mathbb{Z})$ for $s \in B \setminus D$.

Proposition 2.7. *For a stable Lagrangian fibration $f : M \to B$ and $s \in B \setminus D$, fix a generator of the cyclic fundamental group of $\pi_1(B \setminus D, s)$ and denote by $\tau_s : \Lambda_s \to \Lambda_s$ the monodromy operator of the generator. Then the fixed part $\Lambda'_s \subset \Lambda_s$ of τ_s at $s \in B \setminus D$ has corank 1.*

Proof. For any $s \in B \setminus D$, we can identify each fiber $\Lambda_s = H_1(M_s, \mathbb{Z})$ with the fiber Γ_s of Proposition 2.4. Thus the result follows from Proposition 2.5. □

3. Principally polarized stable Lagrangian fibration

Definition 3.1. Let Λ be as in Proposition 2.7. A *principal polarization* on a stable Lagrangian fibration $f : M \to B$ is a unimodular anti-symmetric form $Q : \wedge^2\Lambda \to \mathbb{Z}_{B\setminus D}$ where $\mathbb{Z}_{B\setminus D}$ denotes the constant sheaf of integers on $B \setminus D$, which induces a principal polarization on each smooth fibers of f. A stable Lagrangian fibration with a choice of principal polarization is called a *principally polarized stable Lagrangian fibration*.

Remark 3.2. In Definition 3.1, the polarization on $M \setminus f^{-1}(D)$ may not extend to an f-ample class of the whole M. In fact, f need not be projective. This definition is useful because there are many situations where the polarization exists *a priori* only on the smooth fibers, e.g., in Kodaira's study of elliptic fibrations and also in our construction in Section 5.

Proposition 3.3. *Let $f : M \to B$ be a principally polarized stable Lagrangian fibration. Then the monodromy operator in Proposition 2.7 satisfies $\tau_s \neq \mathrm{Id}$ and $\tau_s \circ \tau_s \neq \mathrm{Id}$.*

Proof. If $\tau_s = \mathrm{Id}$, then we see that f is a smooth fibration, as in the proof of Proposition 3.2 in [Hw]. In fact, since there is no monodromy and f is polarized over $B \setminus D$, we can extend the period map of the abelian family on $B \setminus D$ to the whole B ([Gr], Theorem 9.5). Thus, we obtain a smooth abelian fibration $f' : M' \to B$ such that f and f' are bimeromorphic outside D. Since M' contains no rational curves and both M and M' have trivial canonical bundles, this implies M and M' are biholomorphic, a contradiction to the non-emptiness of the discriminant D of f.

If $\tau_s \circ \tau_s = \mathrm{Id}$, take a double cover $g : B' \to B$ branched along D and let $D' = g^{-1}(D)$. Denote by $\hat{f} : \hat{M} \to B'$ the fiber product of f and g, which has no monodromy on $B' \setminus D'$. By the $\mathbb{C}^{n-1}$-action of Proposition 2.2 which lifts to $\hat{M}$, the following property of $\hat{M}$ can be seen from the corresponding properties in the case of $n = 1$ (cf. Proof of Proposition 9.2 in [BHPV]): $\hat{M}$ is normal, Gorenstein with singularities of type $A_1 \times$ (germ of $(2n-1)$-dimensional manifold) and has trivial canonical bundle. Thus we have a crepant resolution $f' : M' \to B'$, which is a family with trivial canonical bundle and no monodromy. Then we get a contradiction as in the previous case. □

Lemma 3.4. *Let $\tau : \Lambda \to \Lambda$ be an automorphism of a lattice such that $\Lambda' := \{v \in \Lambda, \tau(v) = v\}$ is a sublattice of corank 1, i.e., $\Lambda/\Lambda' \cong \mathbb{Z}$. If $\tau \circ \tau \neq \mathrm{Id}$, then $\eta := \tau - \mathrm{Id}$ satisfies $\eta \circ \eta = 0$.*

Proof. Note that $\Lambda' \subset \mathrm{Ker}(\eta)$. The induced automrophism $\bar{\tau} : \Lambda/\Lambda' \to \Lambda/\Lambda'$ is either Id or $-$Id. If $\bar{\tau} = \mathrm{Id}$, then for a non-zero $v \in \Lambda \setminus \Lambda'$, we have $\tau(v) = v + \lambda$ for some $\lambda \in \Lambda'$. Then $\eta(v) = \lambda \in \Lambda' \subset \mathrm{Ker}(\eta)$. This proves that $\eta \circ \eta = 0$. If $\bar{\tau} = -\mathrm{Id}$, then for a non-zero $v \in \Lambda \setminus \Lambda'$, we have $\tau(v) = -v + \lambda$ for some $\lambda \in \Lambda'$. Then

$$\tau \circ \tau(v) = -\tau(v) + \tau(\lambda) = -(-v + \lambda) + \lambda = v.$$

Thus $\tau \circ \tau = \mathrm{Id}$, a contradiction. □

Proposition 3.5. *In the setting of Proposition 3.3, let $\eta := \tau_s - \mathrm{Id}$. Then for any $\beta \in \mathrm{Im}(\eta)$ and an element $\varphi \in H^1(M, \mathbb{Z})$,*

$$\langle i^*\varphi, \beta \rangle = \langle \varphi, i_*\beta \rangle = 0$$

where $i^* : H^1(M, \mathbb{Z}) \to H^1(M_s, \mathbb{Z})$ and $i_* : H_1(M_s, \mathbb{Z}) \to H_1(M, \mathbb{Z})$ are the homomorphisms induced by the inclusion $i : M_s := f^{-1}(s) \subset M$.

Proof. Let $\mathcal{H}$ be the local system on $B \setminus D$ given by $H^1(M_s, \mathbb{Z}), t \in B \setminus D$. Denote by $\tau^* : \mathcal{H}_s \to \mathcal{H}_s$ the transformation dual to τ, i.e., for any $\varpi \in H^1(M_s, \mathbb{Z})$ and $u \in H_1(M_s, \mathbb{Z})$,

$$\langle \tau^*(\varpi), u \rangle = \langle \varpi, \tau(u) \rangle.$$

By Proposition 2.7, Proposition 3.3 and Lemma 3.4, we have $\eta \neq 0$ and $\eta \circ \eta = 0$, i.e.,

$$0 \neq \mathrm{Im}(\eta) \subset \mathrm{Ker}(\eta) = \Lambda'_s.$$

Similarly, $\eta^* := \tau^* - \mathrm{Id}$ is an endomorphism of $\mathcal{H}_s$ with $\eta^* \neq 0$ and $\eta^* \circ \eta^* = 0$. Since $i^*\varphi \in \mathrm{Ker}(\eta^*)$ by (the easy half of) the global invariant cycles theorem (cf. Theorem 4.24 of [Vo]), for any $\psi \in \mathrm{Ker}(\eta^*)$ and $u \in \Lambda_s$,

$$\langle \psi, \eta(u) \rangle = \langle \eta^*(\psi), u \rangle = 0.$$

It follows that $\langle i^*\varphi, \mathrm{Im}(\eta) \rangle = 0$. □

Remark 3.6. If the family $f : M \to B$ is projective, we could have used the Monodromy Theorem (cf. Theorem 3.15 in [Vo]) in place of Proposition 3.3 and Lemma 3.4 in the above proof. We have used the above approach because we do not want to assume that f is projective.

Proposition 3.7. *For a principally polarized stable Lagrangian fibration $f : M \to B$ and $s \in B \setminus D$, let $\tau_s : \Lambda_s \to \Lambda_s$ be the monodromy operator of Proposition 2.7, which should*

preserve the polarization $Q_s : \wedge^2 \Lambda_s \to \mathbb{Z}$. *Setting* $\eta := \tau_s - \mathrm{Id}$ *as in Proposition 3.5, we have* $\Lambda'_s = \mathrm{Ker}(\eta)$. *Then* $\mathrm{Im}(\eta) \subset \Lambda_s$ *is contained in*

$$\Xi_s := \{v \in \Lambda'_s \,|\, Q(v, w) = 0 \textit{ for all } w \in \Lambda'_s\}.$$

Proof. Since τ_s preserves the polarization Q_s and $\eta \circ \eta = 0$ by Lemma 3.4,

$$Q_s(\eta(v), u) + Q_s(v, \eta(u)) = 0 \text{ for all } v, u \in \Lambda_s.$$

Thus for any $v \in \Lambda_s$ and $u \in \mathrm{Ker}(\eta) = \Lambda'_s$, we have $Q_s(\eta(v), u) = -Q_s(v, \eta(u)) = 0$, which means $\eta(v) \in \Xi_s$. □

Definition 3.8. Let Λ be a free abelian group of rank $2n$. Given a unimodular non-degenerate anti-symmetric form $Q : \wedge^2 \Lambda \to \mathbb{Z}$, a basis $\{p_1, \ldots, p_n, q_1, \ldots, q_n\}$ of Λ is called a *symplectic basis* of Λ with respect to Q if, in terms of the dual basis $\{p^1, \ldots, p^n, q^1, \ldots, q^n\}$ of $\mathrm{Hom}(\Lambda, \mathbb{Z})$,

$$Q = p^1 \wedge q^1 + p^2 \wedge q^2 + \cdots + p^n \wedge q^n.$$

Lemma 3.9. *In the setting of Definition 3.8, let* $\tau : \Lambda \to \Lambda$ *be a group automorphism preserving* Q. *Assume that the subgroup* $\Lambda' \subset \Lambda$ *of elements fixed under* τ *has corank 1. Then there exists a symplectic basis* $\{p_1, \ldots, p_n, q_1, \ldots, q_n\}$ *such that* $\{p_1, \ldots, p_n, q_1, \ldots, q_{n-1}\} \subset \Lambda'$.

Proof. Fix a symplectic basis $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ such that

$$Q = a^1 \wedge b^1 + \cdots + a^n \wedge b^n.$$

The anti-symmetric form $Q|_{\Lambda'}$ must have a kernel of rank 1, i.e.,

$$\Xi := \{v \in \Lambda', \ Q(v, u) = 0 \text{ for all } u \in \Lambda'\}$$

has rank 1. Pick a generator p_1 of Ξ. Since Ξ is primitive, i.e., Λ/Ξ has no torsion, we can write

$$p_1 = \alpha_1 a_1 + \cdots + \alpha_n a_n + \beta_1 b_1 + \cdots + \beta_n b_n$$

with some integers α_i, β_i satisfying $gcd(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n) = 1$. Thus there exists integers $\alpha'_1, \ldots, \alpha'_n, \beta'_1, \ldots, \beta'_n$ such that

$$\alpha'_1 \cdot \alpha_1 + \cdots + \alpha'_n \cdot \alpha_n + \beta'_1 \cdot \beta_1 + \cdots + \beta'_n \cdot \beta_n = 1.$$

Let

$$q_1 := -\beta'_1 a_1 - \cdots - \beta'_n a_n + \alpha'_1 b_1 + \cdots + \alpha'_n b_n.$$

Then $Q(p_1, q_1) = 1$. Define

$$\Lambda'' := \{v \in \Lambda, Q(p_1, v) = 0 = Q(q_1, v)\}.$$

Then $\Lambda'' \subset \Lambda'$ is a lattice of rank $2n - 2$ such that $Q|_{\Lambda''}$ is unimodular and non-degenerate (cf. [GH], the proof of Lemma in p.304). Let $\{p_2, \ldots, p_n, q_2, \ldots, q_n\}$ be a symplectic basis of Λ''. Then $\{p_1, \ldots, p_n, q_1, \ldots, q_n\}$ is a symplectic basis of Λ with the required property. □

Proposition 3.10. *In the setting of Proposition 2.5, identify* $\Lambda_s = H_1(M_s, \mathbb{Z})$ *with* Γ_s *for* $s \in B \setminus D$ *as in the proof of Proposition 2.7. Assume that we have a principal polarization* Q. *Then we can find a collection of components* $\{p_1, \ldots, p_n, q_1, \ldots, q_{n-1}\}$ *of* Γ' *such that for each* $x \in B \setminus D$, *there exists* $q_{n,x} \in \Gamma_x$ *such that* $\{p_{1,x}, \ldots, p_{n,x}, q_{1,x}, \ldots, q_{n,x}\}$ *is a symplectic basis of* $\Lambda_x = \Gamma_x$ *with respect to* Q_x.

Proof. Fix a point $s \in B \setminus D$. The monodromy $\tau_s : \Lambda_s \to \Lambda_s$ preserves the polarization Q_s on Λ_s and fixes $\Lambda'_s = \Gamma'_s$. Applying Lemma 3.9, we have a symplectic basis $\{p_{1,s}, \ldots, p_{n,s}, q_{1,s}, \ldots, q_{n,s}\}$ with $p_{1,s}, \ldots, p_{n,s}, q_{1,s}, \ldots, q_{n-1,s} \in \Lambda'_s$. Since Γ' consists of sections of $g : T^*B \to B$, the vectors $p_{1,s}, \ldots, p_{n,s}, q_{1,s}, \ldots, q_{n-1,s}$ uniquely determine components $p_1, \ldots, p_n, q_1, \ldots, q_{n-1}$ of Λ'. To check the existence of $q_{n,x}$ for any $x \in B \setminus D$, just pick $q_{n,x}$ as any vector in Λ_x contained in the component of Λ containing $q_{n,s}$. $\square$

Proposition 3.11. *In the notation of Proposition 3.10, when $b \in D$, the vector $p_{n,b} \in \Gamma'_b$ regarded as an element of $H_1(Y_o, \mathbb{Z})$ in the notation of Proposition 2.6, lies in the lattice Υ of Proposition 2.6.*

Proof. Suppose not. By our (Assumption) after Definition 2.1, we may assume that M is topologically retractable to $f^{-1}(b)$ and identify $H^1(f^{-1}(b), \mathbb{Z})$ with $H^1(M, \mathbb{Z})$. Then by Proposition 2.6, there exists $\varpi \in H^1(f^{-1}(b), \mathbb{Z}) = H^1(M, \mathbb{Z})$ such that $\langle \varpi, j_* p_{n,b} \rangle \neq 0$. For a point $s \in B \setminus D$, the choice in Proposition 3.10 implies that $p_{n,s} \in \Xi_s$ of Proposition 3.7. Denote by ϖ_s the element in $H^1(M_s, \mathbb{Z})$ induced by $\varpi \in H^1(M, \mathbb{Z})$ under the identification $H^1(f^{-1}(b), \mathbb{Z}) = H^1(M, \mathbb{Z})$. Since $j_* p_{n,b} \in H_1(f^{-1}(b), \mathbb{Z}) = H_1(M, \mathbb{Z})$ and the image of $p_{n,s} \in H_1(M_s, \mathbb{Z})$ in $H_1(M, \mathbb{Z})$ belongs to the same class, Proposition 3.5 and Proposition 3.7 say that

$$\langle \varpi, j_* p_{n,b} \rangle = \langle \varpi_s, p_{n,s} \rangle = 0.$$

This is a contradiction. $\square$

Proposition 3.12. *In Proposition 3.11, the $\mathbb{C}$-linear span of Υ in $T^*_b(B)$ is exactly $\mathbb{C} \cdot dh$ where $h \in \mathcal{O}(B)$ is a defining equation of the divisor D.*

Proof. From the definition of Υ in Proposition 2.6, the linear span of Υ is sent to a fiber of the $\mathbb{C}^*$-bundle. By Definition 2.1 (i), this fiber is a leaf of the characteristic foliation, which is given by the Hamiltonian vector field $\iota_\omega(f^* dh)$ on M. Under the symplecto-morphism Φ in Proposition 2.4, this corresponds to $\mathbb{C} \cdot dh$. $\square$

Proposition 3.13. *Let $\{p_1, \ldots, p_n, q_1, \ldots, q_{n-1}\}$ be as in Proposition 3.10. Then there exists a holomorphic coordinate system $\{z_1, \ldots, z_n\}$ on B such that, regarded as sections of $T^*(B)$,*

$$p_1 = dz_1, \ \ldots, \ p_n = dz_n$$

and D is given by $z_n = 0$.

Proof. Since $p_1, \ldots, p_n$ are closed 1-forms which are point-wise linearly independent at every point of B, we can find coordinates $z_1, \ldots, z_n$ with $p_i = dz_i$. By Proposition 3.12, we may choose z_n to be a defining equation of D. $\square$

Let us recall the classical Riemann condition (e.g. [GH], p.306).

Proposition 3.14. *Let V be a complex vector space of dimension n and let $\Lambda \subset V$ be a lattice of rank $2n$ such that V/Λ is an abelian variety with a principal polarization. For a symplectic basis $\{p_1, \ldots, p_n, q_1, \ldots, q_n\}$ of Λ with respect to the principal polarization $Q : \wedge^2 \Lambda \to \mathbb{Z}$, $\{p_1, \ldots, p_n\}$ becomes a $\mathbb{C}$-basis of V and the period matrix (θ_i^j) defined by*

$$q_i = \sum_{j=1}^{n} \theta_i^j p_j \quad \textit{in } V$$

is symmetric in (i,j) *and* $\mathrm{Im}(\theta_i^j) > 0$.

Theorem 3.15. *Given a principally polarized stable Lagrangian fibration* $f : M \to B$ *with a Lagrangian section* $\Sigma \subset M$*, there exists a holomorphic coordinate system* $(z_1, \ldots, z_n)$ *on* B *such that*

(i) $z_n = 0$ *is a local defining equation of* D*;*

(ii) *on* $B \setminus D$, $dz_1, \ldots, dz_{n-1}, dz_n$ *belong to* Γ' *in the notation of Proposition 2.5;*

(iii) *there exists a symplectic basis* $\{p_{1,s}, \ldots, p_{n,s}, q_{1,s}, \ldots, q_{1,n}\}$ *on each* $\Lambda_s = \Gamma_s, s \in B \setminus D$ *satisfying*

$$p_{1,s} = (dz_1)_s, \ldots, p_{n,s} = (dz_n)_s$$

and the associated period matrix in the sense of Proposition 3.14 is given by

$$\theta_i^j = \frac{\partial^2 \Psi}{\partial z_i \partial z_j} + \frac{\ell}{2\pi\sqrt{-1}} \log z_n$$

for some holomorphic function Ψ *on* B*, which we call a* potential function *of the Lagrangian fibration, and some integer* ℓ*.*

Proof. Let $\{p_1, \ldots, p_n, q_1, \ldots, q_{n-1}\}$ be as in Proposition 3.10 and Proposition 3.13. At a point $s \in B \setminus D$, we add $q_{n,s}$ to get a symplectic basis of Λ_s. By analytic continuation, we get a multi-valued 1-form q_n over $B \setminus D$ such that any choice of a value $q_{n,t}$ of q_n at a point $t \in B \setminus D$, together with $p_{1,t}, \ldots, p_{n,t}, q_{1,t}, \ldots, q_{n-1,t}$, gives a symplectic basis of Λ_t. Using the coordinate system in Proposition 3.13, we can write

$$q_i = \sum_{j=1}^{n} \theta_i^j dz_j,$$

where θ_i^j is a (univalent) holomorphic function on B for each $1 \le i \le n-1$ and $1 \le j \le n$, while θ_n^j is a multi-valued holomorphic function on $B \setminus D$ for each $1 \le j \le n$. By Proposition 3.14, $\theta_j^i = \theta_i^j$ for each $1 \le i, j \le n$. It follows that θ_n^j is univalent holomorphic function on B for each $1 \le j \le n-1$. By the choice of $p_{n,s} \in \Xi_s$ and Proposition 3.7, the monodromy operator $\tau_s : \Lambda_s \to \Lambda_s$ is of the form

$$\tau_s(q_{n,s}) = q_{n,s} + \ell p_{n,s}$$

for some integer ℓ. Thus

$$\tilde{\theta}_n^n \ := \ \theta_n^n - \frac{\ell}{2\pi\sqrt{-1}} \log z_n$$

is also univalent. Set $\tilde{\theta}_i^j := \theta_i^j$ if $(i,j) \neq (n,n)$. Then $\tilde{\theta}_i^j$ is a univalent holomorphic function on B for all values of $1 \le i, j \le n$ and

$$q_i = \sum_{j=1}^{n} \tilde{\theta}_i^j dz_j \text{ for } 1 \le i \le n-1$$

$$q_n = \sum_{j=1}^{n} \tilde{\theta}_n^j dz_j + \frac{\ell}{2\pi\sqrt{-1}} \log z_n \, dz_n.$$

Since q_i's are closed 1-forms on B, we have

$$\frac{\partial \tilde{\theta}_i^j}{\partial z_k} = \frac{\partial \tilde{\theta}_i^k}{\partial z_j} = \frac{\partial \tilde{\theta}_j^i}{\partial z_k}$$

for any $1 \leq i, j, k \leq n$. By Poincaré's lemma, there exists a holomorphic function Ψ such that

$$\tilde{\theta}_i^j = \frac{\partial^2 \Psi}{\partial z_i \partial z_j}.$$

□

4. Construction of principally polarized stable Lagrangian fibrations with given potential functions

In this section, for a sufficiently small n-dimensional polydisk B with coordinate $(z_1, \ldots, z_n)$, we shall construct a principally polarized stable Lagrangian fibration $f : (M, \omega_M) \to B$ with a given potential function $\Psi(z)$. Our construction closely follows Nakamura's toroidal construction [Na]. However, main differences are the following:

(i) the base space B is of dimension n (rather than 1).

(ii) the total space should be not only smooth but also symplectic.

(I) Construction of a non-proper Lagrangian fibration $\tilde{M} \to B$.

For each integer $k \in \mathbb{Z}$, let E_k be a copy of $\mathbb{C} \times \mathbb{C}$ equipped with linear coordinates (x_k, y_k). We define a complex manifold E by identifying points in $\cup_{k \in \mathbb{Z}} E_k$ by the following rule: a point (x_k, y_k) of E_k with $x_k \neq 0$ and $y_k \neq 0$ is identified with a point (x_{k+1}, y_{k+1}) of E_{k+1} with $x_{k+1} \neq 0$ and $y_{k+1} \neq 0$, if and only if

$$x_{k+1} = x_k^2 y_k, \quad \text{and} \quad y_{k+1} = \frac{1}{x_k}.$$

On E, $z_n := x_k y_k$ is a well-defined holomorphic function independent of k and

$$w_n := x_k^{-k+1} y_k^{-k}$$

is a meromorphic function independent of k, with zeros and poles supported on

$$\cup_{k \in \mathbb{Z}} (x_k y_k = 0).$$

Moreover, the 2-forms $dy_k \wedge dx_k$ glue together yielding a holomorphic symplectic form ω_E on E, satisfying

$$\omega_E = dz_n \wedge \frac{dw_n}{w_n}.$$

Fix coordinates

$$(z_1, \ldots, z_{n-1}, w_1, \ldots, w_{n-1})$$

on $\mathbb{C}^{n-1} \times \mathbb{C}^{n-1}$ and regard them as functions on the open subset $\mathbb{C}^{n-1} \times (\mathbb{C}^\times)^{n-1}$ defined by

$$w_1 \neq 0, \ \ldots, \ w_{n-1} \neq 0.$$

Define

$$\tilde{X} := \mathbb{C}^{n-1} \times (\mathbb{C}^\times)^{n-1} \times E.$$

On $\tilde{X}$, we have the holomorphic functions $z_1, \ldots, z_n, w_1, \ldots, w_{n-1}$ and the meromorphic function w_n.

Define a morphism $\tilde{p} : \tilde{X} \to \mathbb{C}^n$ by $(z^1, \dots, z^n)$. The fiber of $\tilde{p}$ over b with $z_n(b) \neq 0$ is isomorphic to

$$(\mathbb{C}^\times)^{n-1} \times \mathbb{C}^\times$$

with coordinates $(w_1, \dots, w_{n-1}, w_n)$ and the fiber over b with $z_n(b) = 0$ is isomorphic to

$$(\mathbb{C}^\times)^{n-1} \times \cup_{k \in \mathbb{Z}} \mathbb{P}^1_k$$

where $\mathbb{P}^1_k$ is a copy of the projective line $\mathbb{P}^1$ with affine coordinate y_k.

We have a holomorphic symplectic 2-form

$$\omega_{\tilde{X}} := \sum_{i=1}^{n-1} dz_i \wedge \frac{dw_i}{w_i} + \omega_E = \sum_{i=1}^{n} dz_i \wedge \frac{dw_i}{w_i}$$

on $\tilde{X}$. From now, we regard $\tilde{X}$ as a symplectic manifold by this symplectic form. From the coordinate expression of $\omega_{\tilde{X}}$ and $\tilde{p}$, it is immediate that $\tilde{p}$ is a *non-proper* Lagrangian fibration.

We denote

$$\tilde{M} = \tilde{X} \times_{\mathbb{C}^n} B$$

where

$$B = \{(z_1, \dots, z_{n-1}, z_n) \,|\, |z_i| < \epsilon (\forall i)\}$$

and ϵ is a sufficiently small positive real number. We denote the natural projection $\tilde{M} \to B$ induced from $\tilde{p}$ by

$$\tilde{f} : \tilde{M} \to B\ .$$

Note that the restriction $\omega_{\tilde{M}}$ of $\omega_{\tilde{X}}$ is a symplectic 2-form on $\tilde{M}$ and $\tilde{f}$ is a non-proper Lagrangian fibration.

(II) Group action of $\Gamma = \mathbb{Z}^n$ on $\tilde{M}$.

Let $\Psi(z_1, z_2, \cdots, z_n)$ be a holomorphic function on B such that the imaginary part $\mathrm{Im}\, \tilde{\theta}(z)$ of the Hessian matrix

$$\tilde{\theta}(z) = (\frac{\partial^2 \Psi}{\partial z_i \partial z_j})$$

is positive definite and ℓ be a positive integer. We define the period matrix $\theta(z)$ by

$$\theta(z) = \tilde{\theta}(z) + \frac{\log z_n}{2\pi\sqrt{-1}} \begin{pmatrix} O_{n-1} & 0 \\ 0 & \ell \end{pmatrix}\ .$$

We will write

$$\tilde{\theta}(z) = \begin{pmatrix} \tilde{\Theta}_1(z) & \tilde{\Theta}_2(z) \\ \tilde{\Theta}_2^t(z) & \tilde{\theta}_n^n(z) \end{pmatrix}\ ,$$

where $\tilde{\Theta}_1(z)$ is the $(n-1) \times (n-1)$ matrix, $\tilde{\Theta}_2(z)$ is the $(n-1) \times 1$ matrix, $\tilde{\Theta}_2^t(z)$ is the transpose of $\tilde{\Theta}_2(z)$ and $\tilde{\theta}_n^n(z)$ is 1×1 matrix.

Set $\Gamma = \mathbb{Z}^{n-1} \oplus \mathbb{Z}$. We define a group action of Γ on $\tilde{M}$ as follows. Let $\gamma = (j, m) \in \Gamma$. Then the action $T_\gamma : \tilde{M} \to \tilde{M}$ is defined in terms of the coordinate functions on $\mathbb{C}^{n-1} \times (\mathbb{C}^\times)^{n-1} \times E_k \subset \tilde{M}$ by

$$T_\gamma^* z_i = z_i \text{ for } i = 1, \dots, n-1$$

$$T_\gamma^*(\Pi_{i=1}^{n-1} w_i^{b_i}) = \exp(2\pi\sqrt{-1}(j\tilde{\Theta}_1(z)b + m\tilde{\Theta}_2^t(z)b)\Pi_{i=1}^{n-1} w_i^{b_i}$$

where $b = (b_i)_{i=1}^{n-1}$ is $(n-1)\times 1$ matrix, and

$$T_\gamma^* x_k = (\exp(2\pi\sqrt{-1}(j\tilde{\Theta}_2(z) + m\tilde{\theta}_n^n(z)))^{-1} x_{k-m\ell}$$

$$T_\gamma^* y_k = \exp(2\pi\sqrt{-1}(j\tilde{\Theta}_2(z) + m\tilde{\theta}_n^n(z)) y_{k-m\ell} \ .$$

It is immediate that $T_\gamma^* T_{\gamma'}^* = T_{\gamma+\gamma'}^*$. Then $T_\gamma \in \mathrm{Aut}\,(\tilde{X}/\mathbb{C}^n)$ and $\gamma \mapsto T_\gamma$ defines an injective group homomorphism from Γ to $\mathrm{Aut}\,(\tilde{M}/\mathbb{C}^n)$. Here $\mathrm{Aut}\,(\tilde{M}/\mathbb{C}^n)$ is the group of automorphisms of $\tilde{M}$ over $\mathbb{C}^n$, i.e., the group of automorphisms g of $\tilde{M}$ such that $\tilde{f}\circ g = \tilde{f}$.

Proposition 4.1. *The action Γ on $\tilde{M}$ is properly discontinuous, free and symplectic, in the sense that $T_\gamma^*\omega_{\tilde{M}} = \omega_{\tilde{M}}$ for each $\gamma \in \Gamma$.*

Proof. Freeness of the action is clear from the description of the action. The proof of proper discontinuity is essentially the same as the proof of [Na], Theorem 2.6. This can be also seen from the concrete description of fibers below in (III), at least fiberwisely.

Let us show that the action is symplectic, i.e., $\omega_{\tilde{M}} = T_\gamma^*\omega_{\tilde{M}}$ for each $\gamma = (j,m)$. This is a new part not considered by [Na]. We have $T_\gamma^* dz_i = dz_i$, $T_\gamma^* dw_i = \exp(2\pi\sqrt{-1}f_i(z))w_i$ for all i, where, in terms of the standard basis $\langle e_i\rangle_{i=1}^{n-1}$ of $\mathbb{C}^{n-1}$,

$$f_i(z) = j\tilde{\Theta}_1(z)e_i + m\tilde{\Theta}_2(z)e_i$$

for $1 \le i \le n-1$ and

$$f_n(z) = j\tilde{\Theta}_2(z) + m\tilde{\theta}_n^n(z) \ .$$

Thus, for i with $1 \le i \le n$, we have

$$T_\gamma^* dz_i = dz_i \ ,$$

$$T_\gamma^* \frac{dw_i}{w_i} = T_\gamma^*(d\log w_i)$$

$$= d\log(T_\gamma^* w_i) = d(2\pi\sqrt{-1}f_i(z)) + d(\log w_i)$$

$$= \frac{dw_i}{w_i} + 2\pi\sqrt{-1}\sum_{k=1}^{n}\frac{\partial f_i}{\partial z_k}dz_k \ .$$

Using these identities, we can compute

$$T_\gamma^*\omega_{\tilde{M}} = \sum_{k=1}^{n} T_\gamma^*(dz_i)\wedge T_\gamma^*(\frac{dw_i}{w_i})$$

$$= \omega_{\tilde{M}} - 2\pi\sqrt{-1}\sum_{i=1}^{n}\sum_{k=1}^{n}\frac{\partial f_i}{\partial z_k}dz_k\wedge dz_i$$

$$= \omega_{\tilde{M}} - 2\pi\sqrt{-1}\sum_{1\le i<k\le n}(\frac{\partial f_i}{\partial z_k} - \frac{\partial f_k}{\partial z_i})dz_k\wedge dz_i \ .$$

On the other hand, by definition of $f_i(z)$ $(1 \le i \le n)$ and definition of $\tilde{\theta}(z)$ from the potential function $\Psi(z)$, we know that

$$f_i(z) = \sum_{\alpha=1}^{n-1} j_\alpha \tilde{\theta}^i_\alpha(z) + m\tilde{\theta}^i_n$$

$$= \sum_{\alpha=1}^{n-1} j_\alpha \frac{\partial^2 \Psi}{\partial z_\alpha \partial z_k} + m\frac{\partial^2 \Psi}{\partial z_n \partial z_k} \ .$$

Since $\Psi(z)$ is holomorphic, it follows that

$$\frac{\partial}{\partial z_k}(\frac{\partial^2 \Psi}{\partial z_\alpha \partial z_i}) = \frac{\partial^3 \Psi}{\partial z_k \partial z_\alpha \partial z_i} = \frac{\partial}{\partial z_i}(\frac{\partial^2 \Psi}{\partial z_\alpha \partial z_k}) \ .$$

Substituting this into the formula above, we obtain that $T_\gamma^* \omega_{\tilde{M}} = \omega_{\tilde{M}}$. □

(III) Group quotient of $\tilde{M}$ by $\Gamma = \mathbb{Z}^n$.

Let $M = \tilde{M}/\Gamma$. By Proposition 4.1, M is a smooth symplectic manifold with symplectic form ω_M induced by $\omega_{\tilde{M}}$ and M admits a fibration $f : M \to B$ induced by $\tilde{f}$. We denote the (scheme theoretic) fiber $f^{-1}(b)$ over $b \in B$ by M_b. Let us describe the fibers M_b.

(III-1) Smooth fibers M_b

First consider the case where $z_n(b) \neq 0$, i.e., the case where M_b is smooth. We have

$$\tilde{M}_b = (\mathbb{C}^\times)^{n-1} \times \{(x_k, y_k) | x_k y_k = b_n\} \simeq (\mathbb{C}^\times)^n_{(w_1, \ldots, w_{n-1}, w_n)} \ .$$

and $w_n = z_n(b)^k y_k$. Let $\langle e_i \rangle_{i=1}^n$ be the ordered standard basis of Γ. From the description in (II), the action of Γ is given by:

$$T_{e_i}^* w_j = \exp(2\pi\sqrt{-1}\theta_i^j(b)) w_j$$

$$T_{e_i}^* w_n = \exp(2\pi\sqrt{-1}\theta_i^n(b)) w_n$$

for $1 \le i \le n-1$ with $\theta_i^j = \tilde{\theta}_i^j$ and

$$T_{e_n}^* w_j = \exp(2\pi\sqrt{-1}\theta_n^j(b)) w_j$$

$$T_{e_n}^* w_n = \exp(2\pi\sqrt{-1}\tilde{\theta}_n^n(b)) z_n(b)^\ell w_n = \exp(2\pi\sqrt{-1}\theta_n^n(b)) w_n \ .$$

Hence $M_b = \tilde{M}_b/\Gamma$ is an n-dimensional principally polarized abelian variety of period $\theta(b)$, as desired. By the description of ω_M, the fibers M_b, $z_n(b) \neq 0$, are also Lagrangian submanifolds.

Proposition 4.2. *For $b \in B, z_n(b) \neq 0$, choose the basis*

$$p_{1,b}, \ldots, p_{n,b}\, q_{1,b}, \ldots, q_{n,b}$$

of $H_1(M_b, \mathbb{Z})$ such that $\tilde{M}_b = \mathbb{C}^n / \langle p_{j,b} \rangle_{j=1}^n$ and

$$q_{i,b} = \sum_{j=1}^n \theta_i^j(b) p_{j,b}$$

for each i $(1 \leq i \leq n)$*. Let*

$$p_b^1, \ldots, p_b^n\, q_b^1, \ldots, q_b^n$$

be the dual basis of $H^1(M_b, \mathbb{Z})$*. Then the integral* 2*-form*

$$L_b := \sum_{i=1}^{n} p_b^i \wedge q_b^i$$

give a monodromy invariant principal polarization of M *over* $B \setminus D$ *where* $D = (z_n = 0)$*.*

Proof. When $z_n(b) \neq 0$, the fiber $\tilde{M}_b$ of $\tilde{p} : \tilde{M} \to B$ is $(\mathbb{C}^\times)^n$ and this family has no monodromy over $B \setminus (z_n = 0)$. Thus we can fix a basis $p_{1,b}, \ldots, p_{n,b}$ of $H_1(\tilde{M}_b, \mathbb{Z})$ uniformly in $b, z_n(b) \neq 0$.

To get a basis of $H_1(M_b, \mathbb{Z})$, we choose additional elements $q_{1,b}, \ldots, q_{n,b} \in H_x(M_b, \mathbb{Z})$ determined by the deck-transformation of $\tilde{M}_b$ induced by the action $T_{e_1}, \ldots, T_{e_n}$. From the description of $T_{e_i}^*$ on w_j, they satisfy the relation

$$q_{i,b} = \sum_{j=1}^{n} \theta_i^j(b) p_{j,b}.$$

We see that $q_{1,b}, \ldots, q_{n-1,b}$ are invariant under the monodromy, while $q_{n,b} \mapsto q_{n,b} + \ell p_{n,b}$ under the monodromy of the generator γ of $\pi_1(B \setminus D)$, i.e., the circle around discriminant divisor $z_n = 0$. The 2-form L_b is a principal polarization on M_b. It remains to show that L_b is invariant under the monodromy. By definition of $\theta(b)$, we compute that

$$\begin{aligned}\gamma^*(L_b) &= \gamma^*(\sum_{i=1}^{n-1} p_{i,b} \wedge q_{i,b}) + \gamma^*(p_{n,b} \wedge q_{n,b}) \\ &= \sum_{i=1}^{n-1} p_b^i \wedge q_b^i + p_b^n \wedge (q_b^n - \ell p_b^n) = L_b.\end{aligned}$$

This implies the invariance. □

Remark 4.3. As in [Na], one can also describe $\tilde{f} : \tilde{M} \to B$ in terms of toric geometry. Following an argument similar to [Na], Section 4, it seems possible to give a relatively principally polarized divisor (the relative theta divsor) which is defined globally over $B \setminus D$. However, its closure is not necessarily f-ample even if total space is of dimension 4 (cases of stable principally polarized Lagrangian 4-folds). In fact, a failure of f-ampleness of the closure already happens when the fiber dimension 2 and the base dimension 1 as explicitly described in [Na] Section 4, Page 219. See also Remark 3.6.

(III-2) Singular fibers M_b

Next consider the singular fibers of f. They are M_b with $z_n(b) = 0$. Recall from (I) that $\tilde{M}_b$ is the product of $(\mathbb{C}^\times)^{n-1}$ with coordinate $(w_i)_{i=1}^{n-1}$ and the infinite tree $\cup_{k \in \mathbb{Z}} \mathbb{P}_k^1$ of projective lines $\mathbb{P}_k^1$ with affine coordinate y_k, and $M_b = \tilde{M}_b / \Gamma$. Let us denote by $(0)_k, (\infty)_k \in \mathbb{P}_k^1$ the two points on the projective line $\mathbb{P}_k^1$ such that $(0)_k$ is identified with $(\infty)_{k-1}$ in the tree.

From (II), the action of Γ is given by:

$$T_{e_i}^* w_j = \exp(2\pi\sqrt{-1}\tilde{\theta}_i^j(b))w_j$$
$$T_{e_i}^* y_k = \exp(2\pi\sqrt{-1}\tilde{\theta}_i^n(b))y_k$$

for $1 \le i \le n-1$ and

$$T_{e_n}^* w_j = \exp(2\pi\sqrt{-1}\tilde{\theta}_n^j(b))w_j$$
$$T_{e_n}^* y_k = \exp(2\pi\sqrt{-1}\theta_n^n(b))y_{k-\ell}\ ,$$

where $\langle e_i\rangle_{i=1}^n$ is the ordered standard basis of Γ. Here we note that the last equality shows that the monodromy operation corresponds to the shift of the components of the infinite tree $\cup_{k\in\mathbb{Z}}\mathbb{P}_k^1$. Thus $\tilde{M}/ < e_n >$ can be described as the variety obtained from

$$(\mathbb{C}^\times)^{n-1} \times \cup_{k=0}^{\ell-1}\mathbb{P}_k^1$$

by identifying the point

$$(w^1, \dots, w^{n-1}) \times (0)_0 \ \in \ (\mathbb{C}^\times)^{n-1} \times (0)_0$$

with the point

$$(\exp(2\pi\sqrt{-1}\theta_n^1)w^1, \dots, \exp(2\pi\sqrt{-1}\theta_n^{n-1})w^{n-1}) \ \in \ (\mathbb{C}^\times)^{n-1} \times (\infty)_{\ell-1}.$$

From this description, M_b consists of ℓ irreducible components, each of whose normalization is isomorphic to a $\mathbb{P}^1$-bundle over $(n-1)$-dimensional complex torus isogenous to $(\mathbb{C}^\times)^{n-1}/\langle e_i\rangle_{i=1}^{n-1}$, where the action of $\langle e_i\rangle_{i=1}^{n-1}$ is given by the coordinate action $T_{e_i}^*$ $(1 \le i \le n-1)$ on w_j $(1 \le j \le n-1)$ described above. Note that the quotient $(\mathbb{C}^\times)^{n-1}/\langle e_i\rangle_{i=1}^{n-1}$ is compact because the imaginary part of $\tilde{\Theta}_1(z)$ is positive definite from the assumption that the imaginary part of $\tilde{\theta}$ is positive definite. We also note that the characteristic cycles are of type I_m for some $1 \le m \le \infty$.

From this description, the following is now clear:

Theorem 4.4. *The fibration $f : M \to B$ constructed above is a proper, flat, principally polarized stable Lagraingian fibration with a potential function $\Psi(z)$. Moreover, ℓ is the number of components of the singular fiber and $S_1^i \neq S_2^i$ for all i in the notation of Definition 2.1.*

Given a principally polarized Lagrangian fibration, we can find a potential function Ψ on B as in Theorem 3.15. Starting from Ψ we can construct a principally polarized stable Lagrangian fibration by Theorem 4.4. These two Lagrangian fibrations must agree outside the discriminant set. Thus they must be biholomorphic by the following.

Proposition 4.5. *Let $f : M \to B$ and $f' : M' \to B$ be two Lagrangian fibrations with the same discriminant $D \subset B$, having Lagrangian sections $\Sigma \subset M$ and $\Sigma' \subset M'$. Suppose there exists a biholomorphic morphism $\Phi : M \setminus f^{-1}(D) \to M' \setminus f'^{-1}(D)$ such that $\Phi(\Sigma) = \Sigma'$ and Φ is symplectomorphic, i.e., $\Phi^*\omega_{M'} = \omega_M$. Then Φ extends to a biholomorphic morphism $M \cong M'$.*

Proof. The proof is essentially given in the proof of Proposition 5.1 in [HO2]. Let us sketch the argument. One can see that Φ is a bimeromorphic map between M and M'. Choose holomorphic coordinates $(z_1, \dots, z_n)$ on B such that the discriminant D is defined by $z_n = 0$. Then the Hamiltonian vector fields induced by $dz_1, \dots, dz_{n-1}$ determine $\mathbb{C}^{n-1}$-actions on

M and M' such that Φ is equivariant with respect to them. These $\mathbb{C}^{n-1}$-actions are free and all orbits have dimension $n-1$. Since both M and M' have trivial canonical bundle, the exceptional loci of the bimeromorphic map must be of codimension ≥ 2. Since they are invariant under the $\mathbb{C}^{n-1}$-actions, they must be union of finitely many orbits of $\mathbb{C}^{n-1}$-actions. But then each component of the exceptional loci in M must be transformed to a component of the exceptional loci in M' biholomorphically. This implies that there are no exceptional loci and Φ is biholomorphic. □

As a corollary of Theorem 4.4 and Proposition 4.5, we obtain

Corollary 4.6. *For any principally polarized stable Lagrangian fibration, the positive integer $|\ell|$ in Theorem 3.15 is the number of components of the singular fiber and $S_1^i \neq S_2^i$ for each i in the notation of Definition 2.1.*

Theorem 3.15, Theorem 4.4 and Corollary 4.6 complete the proof of Theorem 1.1.

5. Periods and the characteristic cycles

In this section, we will examine the relation between types of the characteristic cycles and the periods. For simplicity, we will restrict our discussion to the case of $\ell = 1$. The generalization to arbitrary ℓ is straightforward. Explicit constructions (Constructions I -III) will be given when $n = 2$, i.e., constructions of 4-dimensional principally polarized stable Lagrangian fibrations. Construction I gives an explicit example in which the types of characteristic cycles change fiber by fiber. Construction II gives an explicit example in which the types of characteristic cycles are constant type I_n $(n < \infty)$ and Construction III gives an explicit example in which the types of characteristic cycles are constant type A_∞.

Proposition 5.1. *Let $f : M \to B$ be a $2n$-dimensional principally polarized stable Lagrangian fibration with potential function $\Psi(z)$ and $\ell = 1$. We denote the (univalent) period matrix by $\tilde{\theta}(z) = (\tilde{\theta}_i^j(z))_{i,j=1}^n$ and the multi-valued period matrix $\theta(z)$ of f as*

$$\theta(z) = \tilde{\theta}(z) + \frac{\log z_n}{2\pi\sqrt{-1}} \begin{pmatrix} O_{n-1} & 0 \\ 0 & 1 \end{pmatrix} .$$

For $b \in B$ for which M_b is singular, define $n(b)$ $(1 \leq n(b) \leq \infty)$ to be the order of

$$(\exp(2\pi\sqrt{-1}\tilde{\theta}_n^j(b)))_{j=1}^{n-1} \bmod \langle (\exp(2\pi\sqrt{-1}\tilde{\theta}_i^j(b)))_{j=1}^{n-1} \,|\, 1 \leq i \leq n-1 \rangle$$

in the multiplicative group

$$(\mathbb{C}^\times)^{n-1} / \langle (\exp(2\pi\sqrt{-1}\tilde{\theta}_i^j(b)))_{j=1}^{n-1} \,|\, 1 \leq i \leq n-1 \rangle .$$

Then the characteristic cycle of M_b is of type $I_{n(b)}$.

Remark 5.2. The description above is simpler when $f : M \to B$ is a 4-dimensional principally polarized stable Lagrangian fibration with potential function $\Psi(z) = \Psi(z_1, z_2)$ and $\ell = 1$, as follows. We shall use this description in Constructions (I)-(III) below. We write the (univalent) period matrix $\tilde{\theta}(z)$ and the multi-valued period matrix $\theta(z)$ of f as

$$\tilde{\theta}(z) = \begin{pmatrix} \tilde{\theta}_1(z) & \tilde{\theta}_2(z) \\ \tilde{\theta}_2(z) & \tilde{\theta}_3(z) \end{pmatrix} , \theta(z) = \tilde{\theta}(z) + \frac{\log z_2}{2\pi\sqrt{-1}} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} .$$

For $b = (b_1, 0) \in B$ for which M_b is singular, the characteristic cycle of M_b is then of type $I_{n(b)}$, where $n(b)$ $(1 \le n(b) \le \infty)$ is exactly the order of

$$\exp(2\pi\sqrt{-1}\tilde{\theta}_2(b)) \bmod \langle \exp(2\pi\sqrt{-1}\tilde{\theta}_1(b))\rangle$$

in the multiplicative group $\mathbb{C}^\times/\langle \exp(2\pi\sqrt{-1}\tilde{\theta}_1(b))\rangle$.

Proof. In the description (III-2), M_b is the quotient of $\tilde{M}_b = \cup_{k\in\mathbb{Z}}(\mathbb{C}^\times)^{n-1} \times \mathbb{P}^1_k$ with coordinates $((w_j)_{j=1}^{n-1}, y_k)$ $(k \in \mathbb{Z})$ by the action of $\Gamma = \mathbb{Z}^n$ with ordered standard basis

$$\langle e_1, \ldots, e_{n-1}, e_n\rangle \ .$$

In terms of the standard basis, the action is given by:

$$T_{e_i}^* : (w_j)_{j=1}^{n-1} \mapsto (\exp(2\pi\sqrt{-1}\tilde{\theta}_i^j(b))w_j)_{j=1}^{n-1}$$

$$T_{e_i}^* : y_k \mapsto \exp(2\pi\sqrt{-1}\tilde{\theta}_i^n(b))y_k$$

for $e = i$ $(1 \le i \le n-1)$, and for e_n

$$T_{e_n}^* : (w_j)_{j=1}^{n-1} \mapsto (\exp(2\pi\sqrt{-1}\tilde{\theta}_n^j(b))w_j)_{j=1}^{n-1}$$

$$T_{e_n}^* : y_k \mapsto \exp(2\pi\sqrt{-1}\tilde{\theta}_n^n(b))y_{k-1} \ .$$

Thus $\tilde{M}_b/\langle e_n\rangle$ is $(\mathbb{C}^\times)^{n-1} \times \mathbb{P}^1_0$ in which $((w_j)_{j=1}^{n-1}, 0)$ and $((w'_j)_{j=1}^{n-1}, \infty)$ are identified exactly when the two points $(w_j)_{j=1}^{n-1}$ and $(w'_j)_{j=1}^{n-1}$ of $(\mathbb{C}^\times)^{n-1}$ are in the same orbit under the action of the cyclic subgroup

$$G(b) := \langle (\exp(2\pi\sqrt{-1}\tilde{\theta}_n^j(b)))_{j=1}^{n-1}\rangle$$

of $(\mathbf{C}^\times)^{n-1}$. On the other hand, $((\mathbb{C}^\times)^{n-1} \times \mathbb{P}^1_0)/\langle e_i\rangle_{i=1}^{n-1}$ is the normalization of M_b. Thus, M_b is obtained from $((\mathbb{C}^\times)^{n-1} \times \mathbb{P}^1_0)/\langle e_i\rangle_{i=1}^{n-1}$ by identifying the two $(n-1)-$dimensional complex tori $((\mathbb{C}^\times)^{n-1} \times \{\infty\})/\langle e_i\rangle_{i=1}^{n-1}$ and $((\mathbb{C}^\times)^{n-1} \times \{0\})/\langle e_i\rangle_{i=1}^{n-1}$,by the action of $G(b)$ above. Here, as a subgroup of $(\mathbb{C}^\times)^{n-1}$, the group $\langle e_i\rangle_{i=1}^{n-1}$ is the multiplicative subgroup generated by the $n-1$ elements

$$(\exp(2\pi\sqrt{-1}\tilde{\theta}_i^j(b)))_{j=1}^{n-1} \ , \ 1 \le i \le n-1 \ .$$

This implies the result. $\square$

Construction I.

Under the notation of Section 5, we set $\ell = 1$ and

$$\Psi(z_1, z_2) := \frac{(z_1 + 5\sqrt{-1})^3 + (z_2 + 5\sqrt{-1})^3 + 3z_1^2 z_2 + 3z_1 z_2^2}{6} \ ,$$

Then

$$\tilde{\theta}(z) = \begin{pmatrix} z_1 + z_2 + 5\sqrt{-1} & z_1 + z_2 \\ z_1 + z_2 & z_1 + z_2 + 5\sqrt{-1} \end{pmatrix} , \theta(z) = \tilde{\theta}(z) + \frac{\log z_2}{2\pi\sqrt{-1}} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\mathrm{Im}\,\tilde{\theta}(z) = \begin{pmatrix} y_1 + y_2 + 5 & y_1 + y_2 \\ y_1 + y_2 & y_1 + y_2 + 5 \end{pmatrix} \ .$$

Here and hereafter x_i and y_i are the real and imaginary part of z_i respectively. Since $t+5>0$ and $(t+5)^2-t^2>0$ when $-2<t<2$, it follows that $\operatorname{Im}\theta(z)$ is positive definite on the polydisk

$$\{(z_1,z_2)\,|\,|z_i|<1\}\,.$$

Taking a smaller 2-dimensional polydisk B with multi-radius ϵ, we then obtain a 4-dimensional Lagrangian fibration $f : M \to B$, associated with the potential function $\Psi(z)$ and $\ell = 1$. The discriminant set is $z_2 = 0$. Define $N = N(z)$ to be the order of

$$e^{2\pi\sqrt{-1}z_1} \bmod \langle e^{2\pi\sqrt{-1}(z_1+5\sqrt{-1})}\rangle$$

in the multiplicative group $\mathbb{C}^\times/\langle e^{2\pi\sqrt{-1}(z_1+5\sqrt{-1})}\rangle$. By abuse of language, we include $N=\infty$ when the order is not finite. Then, the characteristic cycle of $M_{(z_1,0)}$ is of type I_N.

Proposition 5.3. *In Construction 1, the characteristic cycle on $M_{(z_1,0)}$ is of Type I_k with $k<\infty$ if and only if*

$$z_1 \in \mathbb{Q}(\sqrt{-1})\ .$$

So, the singular fibers of finite characteristic cycle I_k $(k<\infty)$ and the singular fibers of infinite characteristic cycle I_∞ are both dense over the disciminant set. Moreover, the characteristic cycle of $M_{(z_1,0)}$ is precisely of type I_k $(k<\infty)$ for $z_1 = 1/k$. So, the singular fibers with characteristic cycles of type I_k with any sufficiently large k appear in this family.

Proof. By the definition of $N = N(z)$, it follows that $N < \infty$ for $M_{(z_1,0)}$ if and only if there are integers $k > 0$ and m such that

$$(e^{2\pi\sqrt{-1}z_1})^k = (e^{2\pi\sqrt{-1}(z_1+5\sqrt{-1})})^k\ .$$

The last condition is equivalent to

$$kz_1 - m(z_1+5\sqrt{-1}) \in \mathbb{Z}$$

which is also equivalent to

$$(k-m)x_1 \in \mathbb{Z} \text{ and } (k-m)y_1 - 5m = 0\ .$$

Note that $k-m \neq 0$ in the last equivalent condition, as otherwise $k=m=0$. It is immediate to see that two integers $k>0$ and m satisfying last equivalent condition exist if and only if $z_1 \in \mathbb{Q}(\sqrt{-1})$. Since $|e^{2\pi\sqrt{-1}(z_1+5\sqrt{-1})}| > 1$ for $|z_1|<1$, whereas $|e^{2\pi\sqrt{-1}/k}| = 1$ for $k \in \mathbb{Z}$, it follows that the order $N(z)$ for $z_1 = 1/k$ is precisely the order of $e^{2\pi\sqrt{-1}/k}$ in the multiplicative group $\mathbb{C}^\times$. This implies the last statement. □

Construction II.

Under the notation of Section 5, we set $\ell = 1$ and

$$\Psi(z_1,z_2) := \frac{\sqrt{-1}(z_1^2+z_2^2)}{2} + \frac{z_1z_2}{k}\,,$$

where n is a positive integer. Then

$$\tilde{\theta}(z) = \begin{pmatrix} \sqrt{-1} & 1/k \\ 1/k & \sqrt{-1} \end{pmatrix}, \theta(z) = \tilde{\theta}(z) + \frac{\log z_2}{2\pi\sqrt{-1}} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\operatorname{Im}\tilde{\theta}(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

The matrix $\operatorname{Im}\tilde{\theta}(z)$ is positive definite. So, taking a smaller 2-dimensional polydisk B, we obtain 4-dimensional Lagrangian fibration $f : M \to B$, associated with the potential function $\Psi(z)$ and $\ell = 1$ above. The discriminant set is $z_2 = 0$. The order of

$$e^{2\pi\sqrt{-1}/k} \bmod \langle e^{-2\pi} \rangle$$

is exactly k in the multiplicative group $\mathbb{C}^\times/\langle e^{-2\pi}\rangle$, where $-2\pi = 2\pi\sqrt{-1}\cdot\sqrt{-1}$. Then, the characteristic cycle of $M_{(z_1,0)}$ is of type I_k, and in particular, the type is constant.

Construction III.

Under the notation of Section 5, we set $\ell = 1$ and

$$\Psi(z_1, z_2) := \frac{\sqrt{-1}(z_1^2 + z_2^2)}{2} + \alpha z_1 z_2 ,$$

where α is any irrational, real number, say $\sqrt{2}$. Then

$$\tilde{\theta}(z) = \begin{pmatrix} \sqrt{-1} & \alpha \\ \alpha & \sqrt{-1} \end{pmatrix} , \theta(z) = \tilde{\theta}(z) + \frac{\log z_2}{2\pi\sqrt{-1}} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\operatorname{Im}\tilde{\theta}(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

The matrix $\operatorname{Im}\theta(z)$ is positive definite. So, taking a smaller 2-dimensional polydisk B, we obtain 4-dimensional Lagrangian fibration $f : M \to B$, associated with the potential function $\Psi(z)$ and $\ell = 1$ above. The discriminant set is $z_2 = 0$. Since α is irrational real number, the element

$$e^{2\pi\sqrt{-1}\cdot\alpha} \bmod \langle e^{-2\pi} \rangle$$

is of infinite order in the multiplicative group $\mathbb{C}^\times/\langle e^{-2\pi}\rangle$, where $-2\pi = 2\pi\sqrt{-1}\cdot\sqrt{-1}$. Then, the characteristic cycle of $M_{(z_1,0)}$ is of type A_∞, and in particular the type is constant.

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